The Seven Classes of the Einstein Equations

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Abstract. In current paper we refer to the geometrical classification of the Einstein equations which has been developed by one of the authors of this paper. This classification was based on the classical theory for decomposition of the tensor product of representations into irreducible components, which is studied in the elementary representation theory for orthogonal groups. We return to this result for more detailed investigation of classes of the Einstein equations.

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Introduction

Method of classification of Riemannian manifolds, endowed with tensor structure, on the basis of decomposition of irreducible tensor product of representations (irreducible with respect to the action of orthogonal group) into irreducible components has become traditional in differential geometry (see, e.g., [1] - [5]).

This method is used in theoretical physics (see, for example, [6]; [7]; [8]). For example, in the paper [7] Einstein-Cartan manifolds were classified on the basis of irreducible decomposition of the torsion tensor of an affine-metric connection (irreducible with respect to the action of the Lorentz group). Current results in this modern today research direction in the General Relativity Theory were systemized in paper [7].

At current paper we return back to the work [8] of one of the authors and study in details proposed in this work classification of The Einstein equations on the basis of method mentioned above.

1 The Einstein equations

The space-time (M, g) in the General Relativity Theory is a smooth four-dimensional manifold M, endowed with metrics g of the Lorentz signature (- + + +).

In an arbitrary map (U, φ) with the local coordinate system $\{x^0, x^1, x^2, x^3\}$ the metric g of (M, g) is defined by its components $g_{ij} = g(\partial_i, \partial_j)$, where $\partial_i = \partial/\partial x^i$ for $i, j, k, l, \ldots = 0, 1, 2, 3$. These components define the Christoffel symbols $\Gamma^k_{ij} = 2^{-1}g^{kl}(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$ of the Levi-Civita connection ∇ and the operator of the covariant differention defined by $\nabla_i X^k = \partial_i X^k + \Gamma^j_{il}X^l$ for $X = X^k\partial_k$.

In the General Relativity Theory a metric tensor g is interpreted as gravitational potential and is related by the *Einstein equations*

$$Ric - \frac{1}{2}sg = T : \quad R_{ij} - \frac{1}{2}s\, g_{ij} = T_{ij} \tag{1.1}$$

to mass-energy distribution, which generates the gravitational field. Here Ric is the Ricci tensor of metric g of a space-time (M, g), which can be found from identity $Ric(\partial_i, \partial_j) = R_{ij} = R^k_{ikj}$ for components R^l_{ikj} of torsion tensor R, which, in their turn, can be found from equality $R^k_{lij}X^l = \nabla_i\nabla_j X^k - \nabla_j\nabla_i X^k$, $s := g^{ij}R_{ij}$ – is scalar curvature of metrics g for $(g^{ij}) = (g_{ij})^{-1}$ and, finally, $T_{ij} = T(\partial_i, \partial_j)$ – are components of known *energy-momentum tensor T of matter.*

The curvature tensor R satisfies well-known the Bianchi identities

$$dR = 0 : \quad \nabla_m R^k_{lij} + \nabla_i R^k_{ljm} + \nabla_j R^k_{lmi} = 0. \tag{1.2}$$

From these identities, in particular, following equalities can be found

$$d^*R = -d\, Ric : \quad -\nabla_k R^k_{lij} = \nabla_j R_{li} - \nabla_i R_{lj} = 0, \tag{1.3}$$

which lead to other the Bianchi identities

$$2d^*Ric = -ds : \quad 2g^{kj}\nabla_j R_{ki} = \nabla_i s. \tag{1.4}$$

Equations (1.1) are complemented by "conservation laws"

$$d^*T = 0 : \quad -g^{ik}\nabla_i T_{kj} = 0, \tag{1.5}$$

which are derived from The Einstein equations on the basis of the Bianchi identities (1.4).

2 Invariantly defined seven classes of the Einstein equations

In the paper [8] was proved that on a pseudo-Riemannian manifold (M, g) the bundle $\Omega(M) \subset T^*M \otimes S^2M$, which fiber in each point $x \in M$ consists of linear mappings $\Omega : T_xM \to \mathbf{R}$ such that $\Omega(X, Y, Z) = \Omega(X, Z, Y)$ and $\sum_{i=1}^{n} \Omega(e_i, e_i, X) = 0$ for arbitrary vectors X, Y and Z orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of space T_xM, has point-wise irreducible decomposition (irreducible with respect to the action of pseudo-orthogonal group) $\Omega(M) = \Omega_1(M) \oplus \Omega_2(M) \oplus \Omega_3(M)$.

If as (M, g) we take space-time, then $\nabla T \in \Omega(M)$. As the result six classes of the Einstein equations can be determined via invariant approach Ω_α and $\Omega_\beta \oplus \Omega_\gamma$ $(\alpha, \beta, \gamma = 1, 2, 3 \;\; \beta < \gamma)$, for each of them covariant derivatives ∇T of energy-momentum tensors T of matter are cross-section of relevant invariant subbundles $\Omega_1(M), \Omega_2(M), \Omega_3(M)$ and their direct sums $\Omega_1(M) \oplus \Omega_2(M)$, $\Omega_1(M) \oplus \Omega_3(M)$ $\;\Omega_2(M) \oplus \Omega_3(M)$. Seventh classes determined in [8] satisfy the condition $\nabla T = 0$.

3 The class Ω_1 and integrals of geodesic equations

Class Ω_1 of the Einstein equations is selected via condition (see, [8])

$$\delta^* T = 0 : \quad \nabla_k T_{ij} + \nabla_i T_{jk} + \nabla_j T_{ki} = 0. \tag{3.1}$$

As $trace_g T = -s$, then from (3.1) follows that $d(trace_g T) = 2d^* T = 0$, i.e. $s = const$. In this case equations (3.1) can be re-written in the following form $\delta^* Ric = 0 : \nabla_k R_{ij} + \nabla_i R_{jk} + \nabla_j R_{ki} = 0$. Reverse is evident, i.e. from condition $\delta^* Ric = 0$ can be deduced that $s = const$, and then equations (3.1).

If each solution $x^k = x^k(s)$ of geodesic equations on pseudo-Riemannian manifold (M, g) satisfies condition $a_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = const$ for symmetric tensor field a with local components a_{ij}, then it's said that such geodesic equations admit *first quadratic integral* (see, for example, [9]. In fact, following identity needs to be true $\delta^* a = 0$. At the same time tensor field a, determined by identity above, is called the *Killing tensor* (see [10]). In our case, first quadratic integral of geodesic equations will be $R_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = const$ and Ricci tensor will be consequently the Killing tensor.

Following theorem was proved:

Theorem 1. *The Einstein equations belong to the class Ω_2 if and only if in the space-time (M, g) geodesic lines equations admit following first quadratic integral $R_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = const$ for Ricci tensor $R_{ij} = Ric(\partial_i, \partial_j)$ of metrics g.*

Remark. Let us underline that theory of first integrals equations of geodesic and symmetric tensor Killing fields has many applications in mechanics, general relativity theory and other physics research (see, for example, [10] and [11]).

4 Class Ω_2 and the Young-Mills equations

Class Ω_2 of The Einstein equations is selected via condition (see, [8])

$$dT = 0: \quad \nabla_k T_{ij} = \nabla_i T_{kj}. \tag{4.1}$$

Considering equations (3.1) together with The Einstein equations we get that $s = const$, and consequently we get following equations

$$d\ Ric = 0: \quad \nabla_k R_{ij} = \nabla_i R_{kj}. \tag{4.2}$$

Equations (4.2) are known as *Codazzi equations* (see, for details, [12]). Easy to prove that equations (4.1) follow from Codazzi equations (4.2). In fact, from equations (4.2) follows that $d^* Ric = -ds$. Comparing this equation with Bianchi identities (1.4) we conclude that $s = const$. As the result equations (4.1) will be true. So, equations (4.1) and (4.2) are equivalent.

For constructing an example of space-time (M, g) with $\nabla T \in \Omega_2(M)$, we will take n-dimensional ($n \geq 4$) conformally flat pseudo-Riemannian manifold (M, g), for which as known (see, for example, [9]) following equations are fulfilled

$$d\left[Ric - \frac{1}{2(n-1)} s \cdot g\right] = 0.$$

If we suppose here that $s = const$ we will get equations (4.2). Therefore, on conformally flat space-time (M, g) with the constant scalar curvature s the Einstein equations belong to the class Ω_2.

On arbitrary n-dimensional pseudo-Riemannian manifold (M, g) the *Weyl projective curvature tensor* P is defined by (see, for example, [9]) components $P_{lijk} = R_{lijk} - \frac{1}{n-1}(g_{lj}R_{ik} - g_{lk}R_{ij})$ in arbitrary map (U, φ) with a local coordinate system $\{x^0, x^1, x^2, x^3\}$. For $n > 2$ conversion into zero of tensor P characterises manifold of constant curvature (see [9]). Easy to show that due to (1.3) covariant differential is $d^* P = -\frac{n-2}{n-1} d\ Ric$. We will say that Weyl projective curvature tensor is *a harmonic tensor* if $d^* P = 0$. Definition is explained by the fact that from condition $d^* P = 0$ automatically follows Bianchi identities $dP = 0$. Therefore if tensor P is considered as second form

$P : \Lambda^2(TM) \to \Lambda^2(TM)$, then this form will be simultaneously closed and co-closed and so a harmonic form (see, for example, [13]). Condition for the Weyl projective curvature tensor P to be harmonic leads us to Codazzi equations (4.2), which are equivalent as was proved to (4.1). Following theorem is true:

Theorem 2. *The Einstein equations belong to the class Ω_2 if and only if the Weyl projective curvature tensor P of space-time (M, g) is harmonic.*

It is well known (see, for example, [13]) that connection $\hat{\nabla}$ in main bundle $\pi : E \to M$ over pseudo-Riemannian manifold (M, g) with fiber metrics g_E is called the *Young-Mills field*, if its curvature $\hat{R}$ together with Bianchi identities $d\hat{R} = 0$ satisfies the *Young-Mills equations* $d^*\hat{R} = 0$.

If we consider that $E = TM$, $g_E = g$ and we take Levi-Civita connection ∇ of pseudo-Riemannian metrics g as connection $\hat{\nabla}$, then *Young-Mills equations* $d^*R = 0$ due to equalities (1.3) will become Codazzi equations (4.2), which are equivalent to equations (4.1). Following theorem is true:

Theorem 3. *The Einstein equations belong to the class Ω_2 if and only if the Levi-Civita connection ∇ of metrics g of space-time (M, g) considered as connection in bundle TM is the Young-Mills field.*

Remark. Yang-Mills theory and the other variational theory as Seiberg-Witten theory have been developed greatly and influenced to topology and physics (see [14]; [15]; [16]; [17] and etc), especially in the case of 4-dimensional manifolds. Yang-Mills theory appeared in differential geometry (see, for example, [18], p. 443) as pseudo-Riemannian manifolds with harmonic curvature. This means that pseudo-Riemannian manifolds (M, g) of which curvature tensor R of the Levi-Civita connection ∇ satisfies $d^*R = 0$, i.e. ∇ is a Yang-Mills connection, taking $E = TM$, the tangent bundle of M, and $g_E = g$ as in our pieces 4.

5 The class Ω_3 and geodesic mappings

The class Ω_2 of the Einstein equations is selected via condition (see, [8])

$$\nabla_k R_{ij} = \frac{1}{18} \left(4(\partial_k s) g_{ij} + (\partial_i s) g_{kj} + (\partial_j s) g_{ik}\right). \tag{5.1}$$

Let us recall that diffeomorphism $f : (M, g) \to (\bar{M}, \bar{g})$ of pseudo-Riemannian manifolds of dimension $n \geq 2$ is called *geodesic mapping* (see [19], p. 70), if each geodesic curve of manifold (M, g) is transferred via this mapping onto geodesic curve of manifold $(\bar{M}, \bar{g})$.

An arbitrary $(n+1)$-dimensional $(n \geq 1)$ pseudo-Riemannian manifold (M, g) which scalar curvature $s \neq const$ and the Ricci tensor Ric satisfies the

following equations:

$$\nabla_k R_{ij} = \frac{n-2}{2(n-1)(n+2)} \left(\frac{2n}{n-2} (\partial_k s) g_{ij} + (\partial_i s) g_{kj} + (\partial_j s) g_{ik} \right) \tag{5.2}$$

for a local coordinate system $\{x^0, x^1, \ldots, x^n\}$ is called *a manifolds* L_{n+1}. These manifolds were introduced by N.S. Sinyukov (see [19], pp. 131-132). Manifold L_{n+1} are an example of pseudo-Riemannian manifolds with non-constant curvature which admit *non-trivial geodesic mappings*. Easy to check that for $n = 3$ equations (5.1) and (5.2) are equivalent and that is why spaces-times (M, g) for which $T \in \Omega_3$ are *Sinyukov spaces* L_{n+1} and due to this reason admit nontrivial geodesic mappings.

Following theorem is true:

Theorem 4. *The Einstein equations belong to the class Ω_2 if and only if the space-time (M, g) is a Sinyukov space L_4.*

In addition, we call that if (M, g) is an $(n+1)$-dimensional Sinyukov manifold then the metric form ds^2 of the manifold (M, g) has the following form (see [19], pp. 111-117 and [20])

$$ds^2 = g_{00}(x^0)(dx^0)^2 + f(x^0) \sum_{a,b=1}^{n} g_{ab}(x^1, \ldots, x^n) dx^a dx^b \tag{5.3}$$

for special local coordinate system $\{x^0, x^1, \ldots, x^n\}$. The function $f(x^0)$, components $g_{00}(x^0)$ and $g_{ab}(x^1, \ldots, x^n)$ of the metric form (5.3) satisfy some defined properties which can be found in the paper [20]. From this we conclude that a space-time (M, g) with the Einstein equations of the class Ω_2 is a warped product space-time and the Robertson-Wolker space-time, in particular (see, for example, [21]).

6 Three other classes

In brief we will describe remaining three classes of the Einstein equations. At the beginning lets consider class of the Einstein equations $\Omega_1 \oplus \Omega_2$ defined by the following condition $g^{ij} \nabla_k T_{ij} = 0$ or equivalent condition $s = const$. Following theorem is true:

Theorem 5. *The Einstein equations belong to the class $\Omega_1 \oplus \Omega_2$ if and only if the metric g of the space-time (M, g) has constant scalar curvature.*

Next class of the Einstein equations $\Omega_2 \oplus \Omega_3$ is defined by the following condition $d \left[T - \frac{1}{3} (trace_g T) g \right] = 0$. To obtain its analytical characterisation lets use the *Weyl tensor* W of conformal curvature (see, for example, [9]). It

is known that on arbitrary pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$ this tensor follows equation (see, for example, [9] and [18])

$$d^*W = -\frac{n-3}{n-2} d \left[Ric - \frac{1}{2(n-1)} s \cdot g \right]. \tag{6.1}$$

We'll say that the Weyl tensor of conformal curvature is a *harmonic tensor* if $d^*W = 0$ (see also [18]). Definition is explained by the fact that from condition $d^*W = 0$ (see [9]) automatically follows Bianchi identities $dW = 0$. Therefore if the Weyl tensor W of conformal curvature is considered as second form $W : \Lambda^2(TM) \to \Lambda^2(TM)$, then this form will be simultaneously closed and coclosed and so harmonic (see [13]).

If the equation $T - \frac{1}{3}(trace_g T)g = Ric - \frac{1}{6} s \cdot g$ is true, then for space-time (M, g) due to (6.1) condition $\nabla T \in \Omega_2(M) \oplus \Omega_3(M)$ is equivalent to harmonicity requirement $d^*W = 0$ for the Weyl tensor W of conformal curvature.

Following was proved:

Theorem 6. *The Einstein equations belong to the class $\Omega_2 \oplus \Omega_3$ if and only if the tensor W of conformal curvature of the space-time (M, g) is harmonic.*

The example of a space-time (M, g) with $\nabla T \in \Omega_2(M) \oplus \Omega_3(M)$ is a conformally flat space-time (M, g), this means that $W \equiv 0$ (see [9], p 116) and, consequently, condition $d^*W = 0$ is fulfilled automatically.

The last, third, class $\Omega_1 \oplus \Omega_3$ is defined by the following condition $\delta^* \left[T - \frac{1}{6}(trace_g T)g \right] = 0$. On the basis of the identity $T - \frac{1}{6}(trace_g T)g = Ric - \frac{1}{3} s \cdot g$, our condition turns into following differential equations

$$\delta^*(Ric - \frac{1}{3} s \cdot g) = 0 :$$

$$\nabla_k R_{ij} + \nabla_i R_{jk} + \nabla_j R_{ki} = \frac{1}{3} \left((\partial_k s) g_{ij} + (\partial_i s) g_{jk} + (\partial_j s) g_{ki} \right). \tag{6.2}$$

In this case $R_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = const$ will be first quadratic integral of light-like geodesic equations $x^k = x^k(s)$ of the space-time (M, g). It is obvious that on a Sinyukov manifold L_4 equations (6.2) become identity.

Remark. The seventh class of the Einstein equations is defined by the property $\nabla T = 0$. Chaki and Ray have studied a space-time with covariant-constant energy-momentum tensor T (see [22]). In addition, two particular types of such space-times were considered and the nature of each was determined (see also [23]).

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